Compressed Sensing Neyman-Pearson Based Activity Detection for Sparse Multiuser Communications

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***Abstract*—Compressed Sensing Multiuser Detection (CS-MUD) is a novel approach towards decreased signalling overhead in Massive Machine Type Communications (MMC). At its core CS-MUD employs techniques from Compressed Sensing and sparse signal processing to detect activity and data of nodes in MMC directly at the PHY. This approach inherently requires proper control of errors occurring at the activity detection as false alarm and missed detection errors. These two error events heavily impact the system performance and also impact the performance of higher layer error handling techniques. To simplify handling of activity errors, this paper introduces a new algorithm to keep one of these error rates constant over the whole SNR while minimizing the counterpart. This approach is commonly known as Neyman-Pearson detection and allows for detection at constant false alarm or constant missed detection rate. To employ Neyman-Pearson detection in CS-MUD we introduce a so called adaptive threshold Neyman-Pearson approach which is based on activity Log Likelihood ratios.**

***Index Terms*—Compressed Sensing; Multiuser Detection; Sparse Signal Processing; Neyman-Pearson Detection**

I. Introduction

Massive Machine Type Communications (MMC) is one of the big emerging challenges raising new demands on todays communication networks. The term MMC refers to setups where a large number of nodes sporadically transmit small data packets to a central aggregation point. Applying an access reservation protocol, e.g., like in LTE for the coordination of node access would require an significant amount of additional transmissions between the massive number of nodes and the aggregation point, making the transmission of small data packets inefficient. To decrease this overhead direct random access can be applied. The downside of this approach are collisions of packets, especially in massive MMC setups. As a solution, recent research has been focused on multi-packet reception or multiuser detection (MUD) at the PHY layer for resolving collisions in random access transmissions. Here the PHY layer detection task is to identify the activity of the nodes and detect data only for active nodes. A recently developed scheme for the detection in MMC is Compressed Sensing Multiuser Detection (CS-MUD) [1], [2]. CS-MUD is a framework that enables the joint detection of data and node activity in highly overloaded MMC setups where only a few observations are available at the aggregation node. The novelty of this approach is to perform activity detection via Compresses Sensing inspired algorithms by interpreting inactive nodes as transmitting zeros instead of modulation symbols [3]–[5]. With this interpretation the activity detection task is to identify the non-zero elements in a sparse multiuser signal containing many zeros. Recent works have shown different concepts for CS-MUD and it was demonstrated that CS-MUD is able to nearly achieve the same performance as a system with access reservation techniques [6].

The application of CS-MUD techniques requires novel concepts for the control of activity error events. Due to the activity detection task, CS-MUD can erroneously detect inactive nodes as active and active nodes as inactive. These two error events are commonly known as false alarm and missed detection errors and heavily impact the system performance. In case of missed detection errors, data of that particular node is lost and cannot be recovered. False alarm errors lead to so called pseudo data that can be identified and discarded at higher layers. However, Automatic Repeat Request (ARQ) schemes suffer from high false alarm rates as they assume erroneous reception of transmitted packets, which is not the case for false alarm errors, where the corresponding node did not send anything. These two error events are not explicitly considered in Compressed Sensing algorithms, but have to be taken into regard and controlled for possible system requirements.

Recent works by the authors has focused on balancing the false alarm and missed detection errors by so called Bayes-Risk based detection [2]. However, these concepts only allow a balancing between both error rates without giving guarantees for the maximum error rates.

This paper focuses at algorithms for detection at constant false alarm or missed detection rate that does not change over the whole SNR and therefore eases the adaptation to given system requirements. Thus, we utilize the concept of Neyman-Pearson [7] estimation to design a constant false alarm or missed detection rate detector. This technique guarantees that the probability of one type of error event (false alarm or missed detection) is kept below a pre-defined threshold. Based on some statistical knowledge, the key for Neyman-Pearson detection is to optimize the decision threshold between both underlying hypothesis (active / inactive) such that pre-defined error probabilities are not exceeded. This approach converts the detection problem into a decision threshold finding problem.

Finding this optimal threshold in a CS-MUD setting is not straight forward due to the lack of analytic descriptions of the false alarm and missed detection rates. However, we make use of *activity* Log-Likelihood ratios (LLRs) which allows us to estimate the false alarm and missed detection rate adaptively. This approach enables the calculation of an optimal threshold for a given set of activity LLRs adaptively. To demonstrate the feasibility of this approach we investigate the error rates in an MTC uplink setup.

II. Detection Model

In the following we present a multiuser uplink system model also depicted in Fig. 1. K nodes sporadically access the wireless channel to transmit data to a central aggregation node. To capture the sporadic nature of the nodes, we assume that each node is active with probability p_a, which is assumed to be identical for all nodes in the system. Augmenting our concepts for individual activity probabilities is straight forward and omitted to simplify notation. Active nodes spread their modulation symbols prior to transmission via Gaussian spreading sequences of length m. Gaussian sequences have shown good reconstruction properties in the Compressed Sensing context, mainly driven by their property of tightly fulfilling the Restricted Isometry Property (RIP) [8]. Collecting the modulation symbols from all K nodes in a vector yields the multiuser vector $\mathbf{x} \in \mathcal{A}_0^K$, where $\mathcal{A}_0 = \mathcal{A} \cup \{0\}$ denotes the modulation alphabet used by the nodes $\mathcal{A}$ augmented by the zero symbol to capture node inactivity. Collecting the node specific Gaussian spreading sequences in a matrix $\mathbf{T} \in \mathbb{C}^{m \times K}$ allows us to state the following detection model for time instance i.

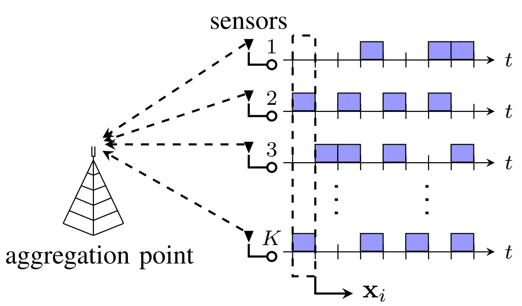

Fig. 1. Sporadic Machine-to-Machine uplink scenario with K nodes transmitting to a single aggregation node.

$$\mathbf{y}_i = \mathbf{T}\mathbf{x}_i + \mathbf{n}_i. \tag{1}$$

Here $\mathbf{n}_i \sim \mathcal{N}\left(0, \sigma_n^2 \mathbf{I}_m\right)$ denotes the uncorrelated white Gaussian noise with variance σ_n^2. In cases where $m < K$, the system is called overloaded as the number of resources (here spreading sequence length) is lower than the number of users. Subsequently, we mainly focus on overloaded systems. The detection model (1) implicitly assumes synchronous reception and perfect channel state information, which is not given in general. Previous research on CS-MUD has demonstrated concepts for detecting unknown delays or channel coefficients in addition to the activity and the data [9], [10]. In this work we mainly focus on the activity detection task and assume an AWGN channel with also perfect delay estimation, which can either be obtained via periodic channel sounding or CS-MUD related techniques that have been investigated in [9], [10].

III. Signal Reconstruction

Classical CS techniques for under-determined noisy systems like that one in (1) usually utilize a sparsity promoting l_1-norm as a convex relaxation [11] of the l_0-"norm"[1] to set up a convex optimization problem. The most prominent candidate of such formulation is the Basis Pursuit De-noising (BPDN) [12] which minimizes an l_1 cost function with respect to a least-squares constraint on the residual. Another class of algorithm is the class of so called greedy algorithms. Greedy approaches aim at solving the Compressed Sensing problem iteratively. Here the Orthogonal Matching Pursuit and Orthogonal Least Squared algorithms seems to be the most prominent candidates for greedy approaches [13]. However, these two approaches already demonstrate that Compressed Sensing algorithms lack any type false alarm or missed detection control complicating the application to systems with asymmetric consequences to false alarm / missed detection errors as described previously.

A. Neyman-Pearson Detection

One approach for controlling false alarm and missed detection errors is to minimize one type of error class (false alarm / missed detection) while bounding the probability for the counterpart [7]. This technique is known as Neyman-Pearson based detection. To keep clarity, we subsequently focus on minimization of the false alarm rate while bounding the missed detection rate. As an example, assume a scenario where based on an observation y, a binary likelihood ratio test decides between the two hypothesis H_I : **Inactive** with likelihood p_I and counter-hypothesis H_A : **Active** with likelihood p_A. The likelihood ratio reads

$$\frac{p_I\left(y\right)}{p_A\left(y\right)} \underset{H_A}{\overset{H_I}{\gtrless}} t, \tag{2}$$

where t is the decision threshold between deciding in favor of H_I or H_A. The set of outcomes where (2) decides in favor of inactive is parametrized by t and reads $R_I\left(t\right) := \{y : p_I\left(y\right) > t p_A\left(y\right)\}$. Subsequently, the missed detection probability depends now on t and reads $\mathrm{Pr}_{\mathrm{Md}}\left(t\right) = \int_{R_I(t)} p_A\left(y\right) dy$. The same functional relationship can be expressed for the false alarm probability. The aim of a Neyman-Pearson test is to find the optimal threshold $t^\star$ such that the false alarm probability $\mathrm{Pr}_{\mathrm{Fa}}\left(t\right)$ is minimized while the counterpart is kept below a certain pre-defined value η.

$$t^\star = \arg\min \mathrm{Pr}_{\mathrm{Fa}}\left(t\right) \quad \text{s.t. } \mathrm{Pr}_{\mathrm{Md}}\left(t\right) \leq \eta, \tag{3}$$

[1]In fact the zero-"norm" is neither a norm nor a pseudo norm. However, the expression zero-"norm" is commonly used in Compressed Sensing contexts

Thus the goal for fulfilling (3) is to find $t^\star$ which can be obtained if the error probabilities $\Pr_{\text{Md}}(t)$ and $\Pr_{\text{Fa}}(t)$ can be expressed in closed form. For the model given in (1), closed form expressions for missed detection and false alarm probabilities are hard to formulate which is due to the multiuser problem, where each element in $\mathbf{y}$ contains information about the whole multiuser vector $\mathbf{x}$.

B. Activity LLRs

Our approach to approximately calculate $\Pr_{\text{Md}}(t)$ and $\Pr_{\text{Fa}}(t)$ for (1) is based on the estimation via activity LLRs for the elements x_k. These activity LLRs indicate the likelihood that x_k was either zero or non-zero. As shown later the activity LLRs allow us to infer the false alarm and missed detection probability.

We define the activity LLRs for the kth element as $L_k = \log\left(\Pr\left(x_k = 0\right)/\Pr\left(x_k \in \mathcal{A}\right)\right)$. Moreover, we infer the value of x_k in a Bayesian framework from the observation yielding $\Pr(x_k) := \Pr(x_k|\mathbf{y})$. To formulate $\Pr(x_k|\mathbf{y})$ we have to marginalize the joint distribution of all possible source vectors with respect to the kth element. Therefore, the set of all K dimensional vectors with elements from the set $\mathcal{A}_0$ where the kth element is equal to ν is denoted as $\mathbb{X}_{x_k=\nu} = \{\mathbf{x} \in \mathcal{A}_0^K : x_k = \nu\}$. The activity LLR can then be formulated via a Generalized Likelihood Ratio Test (GLRT) [14] yielding

$$L_k = \log \frac{\sum_{\mathbf{x}\in\mathbb{X}_{x_k=0}} p(\mathbf{y}|\mathbf{x}) \Pr(\mathbf{x})}{\sum_{\mathbf{x}\in\mathbb{X}_{x_k\in\mathcal{A}}} p(\mathbf{y}|\mathbf{x}) \Pr(\mathbf{x})}, \tag{4}$$

where the hypothesis that x_k is active (denominator) is composite and has to be calculated over the alphabet $\mathcal{A}$. With assumption of Gaussian noise, the likelihood function can be found as

$$p(\mathbf{y}|\mathbf{x}) \propto \exp\left(-\frac{1}{2\sigma_n^2}\|\mathbf{y} - \mathbf{T}\mathbf{x}\|_2^2\right). \tag{5}$$

Considering (4) the calculation of a single activity LLR involves a summation over nearly $|\mathcal{A}^K|$, which is clearly infeasible if K is large. To combat this issue, we make use of the fact that the summation is over exponential functions and utilize the so called max-log approximation outlined in [14]. This turns the evaluation of the sums into the two optimization problems

$$\begin{aligned} L_k \approx &\min_{\mathbf{x}\in\mathbb{X}_{x_k\in\mathcal{A}}} \left[\frac{1}{2\sigma_n^2}\|\mathbf{y} - \mathbf{T}\mathbf{x}\|_2^2 - \log(\Pr(\mathbf{x}))\right] - \\ &\min_{\mathbf{x}\in\mathbb{X}_{x_k=0}} \left[\frac{1}{2\sigma_n^2}\|\mathbf{y} - \mathbf{T}\mathbf{x}\|_2^2 - \log(\Pr(\mathbf{x}))\right]. \end{aligned} \tag{6}$$

The a-priori probability for the source vector can be found using the assumptions about node (in)activity made in Section II, determining the probabilities of the elements in $\mathbf{x}$

$$\begin{aligned} \log(\Pr(\mathbf{x})) &= \log\left((1-p_a)^{K-\|\mathbf{x}\|_0}\left(\frac{p_a}{|\mathcal{A}|}\right)^{\|\mathbf{x}\|_0}\right) \\ \log(\Pr(\mathbf{x})) &\propto -\|\mathbf{x}\|_0 \log\left(\frac{1-p_a}{p_a/|\mathcal{A}|}\right). \end{aligned} \tag{7}$$

Here, $\|\mathbf{x}\|_0$ is the zero-"norm" and counts the number of non-zero elements contained in $\mathbf{x}$ [11]. Inserting the prior probability (7) into (6) finally yields the approximated activity LLRs

$$\begin{aligned} &L_k \approx \\ &\min_{\mathbf{x}\in\mathbb{X}_{x_k\in\mathcal{A}}} \left[\frac{1}{2\sigma_n^2}\|\mathbf{y} - \mathbf{T}\mathbf{x}\|_2^2 + \|\mathbf{x}\|_0 \log\left(\frac{1-p_a}{p_a/|\mathcal{A}|}\right)\right] \\ &- \min_{\mathbf{x}\in\mathbb{X}_{x_k=0}} \left[\frac{1}{2\sigma_n^2}\|\mathbf{y} - \mathbf{T}\mathbf{x}\|_2^2 + \|\mathbf{x}\|_0 \log\left(\frac{1-p_a}{p_a/|\mathcal{A}|}\right)\right]. \end{aligned} \tag{8}$$

C. Adaptive Threshold Neyman-Pearson Detection

Given a certain decision threshold, the activity LLRs defined in (8) allow for a very simple estimator of the false alarm and missed detection probabilities. Recalling that $L(x_k) = \log\frac{\Pr(x_k=0)}{\Pr(x_k\in\mathcal{A})}$ allows to estimate the posteriori probabilities from the LLRs via [15]

$$\Pr(x_k \in \mathcal{A}|L_k) = \frac{1}{1+\exp L_k} \tag{9a}$$

$$\Pr(x_k = 0|L_k) = \frac{1}{1+\exp(-L_k)}, \tag{9b}$$

i.e., the LLRs can be used to infer the posteriori probability that each hypothesis is the correct one. To calculate the false alarm and missed detection probabilities, a threshold t and a mapping rule ϕ for the activity LLRs is required. The following mapping rule is applied in the sequel of this work

$$\phi(x_k, t) = \begin{cases} 0 & \text{if} \quad L_k \geq t \\ 1 & \text{if} \quad L_k < t. \end{cases} \tag{10}$$

Here the element $\hat{x}_k = 0$ if the node is estimated as inactive and $\hat{x}_k = 1$ if the node is estimated as active. Consequently, the output of this estimator is the set of active nodes that can be forwarded to the data detector. In this paper we only consider activity detection. Subsequent data detection can be facilitated by known techniques from communications.

Given the activity LLRs, the following theorem states how the estimation of false alarm and missed detection probabilities is achieved.

Theorem 1. *Given a set of activity LLRs of cardinality* $N_L = |\mathcal{L}|$ *with* $L_k \in \mathbb{R}$, $k = 1, \cdots, N_L$, *a decision threshold* $t \in \mathcal{L}$ *and a decision rule* $\phi : L_k \mapsto \{0,1\}$ *that divides* $\mathcal{L}$ *into the two subsets* $\mathcal{L}_A = \{k : L_k < t\}$ *and* $\mathcal{L}_I = \{k : L_k \geq t\}$. *The set specific false alarm and missed detection probabilities can be approximated via*

$$\Pr_{\text{Md}}(\mathcal{L}, t) \approx \frac{\sum_{\mathcal{L}_i} \Pr(x_k \in \mathcal{A}|L_k)}{|\mathcal{L}_I|} \tag{11a}$$

$$\Pr_{\text{Fa}}(\mathcal{L}, t) \approx \frac{\sum_{\mathcal{L}_a} \Pr(x_k = 0|L_k)}{|\mathcal{L}_A|}, \tag{11b}$$

which converge in the limit to the true false alarm and missed detection probabilities.

The proof of this theorem is given in Appendix A. The formulation in (11) states that fixing t to any value, separates the calculated activity LLRs into two sets. $\mathcal{L}_I$ contains the indices of elements that are estimated to be inactive whereas $\mathcal{L}_a$ contains the complementary index set. A missed detection error occurs if the kth element is contained in $\mathcal{L}_I$ while the kth entry in the source vector was non-zero originally. The posteriori probabilities for such *wrong* decisions can be calculated via (9). Averaging over these probabilities in a set gives an estimate of the false alarm and missed detection probability for a set of activity LLRs $\mathcal{L}$. The set specific false alarm or missed detection probability allows to find the set specific threshold $t^\star_{\mathcal{L}}$ such that the set specific false alarm or missed detection probability is minimized while bounding the other one to an upper limit η. The optimal set specific decision threshold can be found analogous to (3) via

$$t^\star_{\mathcal{L}} = \arg\min \Pr_{\text{Fa}}(\mathcal{L}, t) \quad (12)$$

$$\text{s.t. } \Pr_{\text{Md}}(\mathcal{L}, t) \leq \eta, \quad (13)$$

where η is the desired target error rate that must not be exceeded. Investigating objective and constraint in (12) as functions of t shows, that $\Pr_{\text{Md}}(\mathcal{L}, t)$ is strictly decreasing whereas $\Pr_{\text{Fa}}(\mathcal{L}, t)$ is strictly increasing. Additionally, t can only take values from the set $\mathcal{L}$. Consequently, $t^\star$ is the largest value t that still fulfills and maybe even over-fulfills the constraint. One way to solve this optimization problem is to try all possible $t \in \mathcal{L}$ until the optimal value has been attained. As this search is scalar and over a restricted set, the computational burden of trying each possible $t \in \mathcal{L}$ are low.

As shown in Appendix A, the precision of this approach depends on the sample size N_L. Thus, a simple way of enhancing the precision is to increase the sample size N_L of activity LLRs. For example, consecutive measurements can be processed jointly to obtain a common activity estimate. In the context of a multiuser uplink system this refers to activity estimation on a per frame basis, i.e. nodes are (in)active for the duration of a whole frame of L symbols. In this case multiple observations $\mathbf{y}_i, 1 \leq i \leq L$ can be processed jointly to produce $N_L = LK$ activity LLRs. Such consecutive processing allows to capture the statistics of the activity LLRs properly. The amount of activity LLRs that is necessary such that (12) can be solved accurately is dependent on the desired threshold and on the statistical properties of the activity LLRs.

IV. Performance Evaluation

A. Implementation Aspects

In the following we demonstrate the performance of the Neyman-Pearson activity LLR estimator on a multiuser uplink system. Before describing the setup, we briefly sketch a possible implementation of such an estimator. The described detection procedure requires the activity LLRs estimation prior to any further processing. Therefore one has to solve (8) which is a complex problem due to the use of the l_0 "norm" and the restriction to finite alphabets. Sphere Decoding [16] has been identified as a good candidate for solving these type of problems efficiently. The detailed description of how (8) can be cast such that solutions via Sphere Decoding are possible is outlined in [2]. Up to now, there exists no polynomial upper bound for the complexity of Sphere Decoding and complexity can only be bounded to be exponential on the size of the problem. However, simulations have shown that Sphere Decoding can outperform polynomial algorithms in terms of complexity especially in the high SNR regime [17]. In this paper we use Sphere Decoding for calculating the activity LLRs, however, we do not give a complexity analysis and refer the reader to [17]. Additionally, suboptimal algorithms such as Successive Interference Cancellation or K-Best detection have shown to nearly reach the performance of Sphere Decoding in polynomial time. However, in this paper we want to demonstrate the performance of the Neyman-Pearson based approach without discussing suboptimal algorithms and leave implementation aspects subject to further research.

B. Setup

We assume an overloaded multiuser uplink setup analogous to the description in Section II involving $K = 20$ nodes that sporadically access the wireless channel to transmit data to a central aggregation point. The remaining simulation parameters are summarized in Table I. We start by investigating the

Number of Nodes	$K = 20$
Spreading seq. length	$m = 5$
Per Element activity probability	$p_a = 0.2$
Channel / Noise Type	AWGN
Signal Alphabet	$\mathcal{A} = \{\pm 1\}$

TABLE I
Simulation Parameter

false alarm and missed detection rates for the system outlined previously. The target missed detection rates are exemplary set to $\eta_1 = 0.1$ and $\eta_2 = 0.01$. The number of observations that are processed simultaneously is set to $L_1 = 10$ for $\eta_1 = 0.1$ and $L_2 = 100$ for $\eta_2 = 0.01$, respectively. The optimal number of simultaneously processed observation is subject to some heuristics and there is no distinct selection rule. In our simulations we observed that $\frac{1}{\eta}$ is a good choice for capturing the statistics of the system properly. Note that overestimating L does not harm in this case.

Figure 2 shows the false alarm and missed detection rates over the inverse noise variance. Considering the curves for $\eta_1 = 0.1$ shows that the missed detection rate is nearly constant as the noise power changes while the false alarm rate decreases in the higher SNR regime. The pre-defined threshold of $\eta_1 = 0.1$ is slightly over-fulfilled over the whole SNR range. Figure 2 also shows the error rates for the missed detection constraint of $\eta_2 = 0.01$. It can be seen that this constraint is highly over-fulfilled in the low SNR range and approximately fulfilled at higher SNR ranges. It is commonly known that LLRs yield a low mean value at low SNR values due to the high noise variance. As shown in Appendix A the mean value of the activity LLR impacts the error of the adaptive threshold Neyman-Pearson approach.

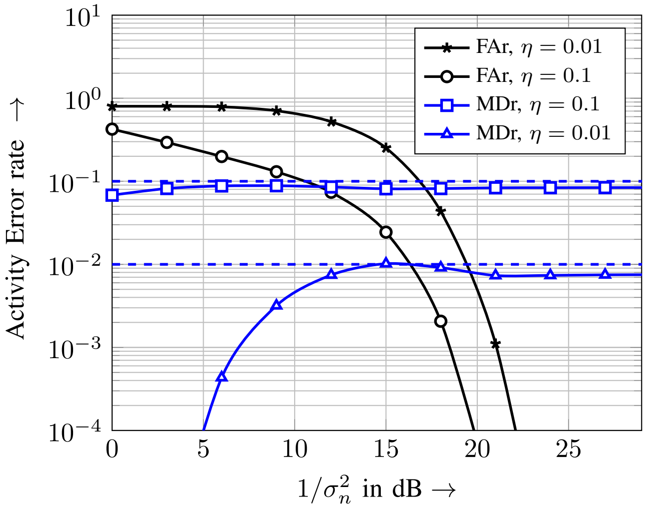

Fig. 2. False alarm (FAr) and missed detection (MDr) rates for adaptive threshold Neyman-Pearson detector with missed detection thresholds $\eta_1 = 10^{-1}$ and $\eta_2 = 10^{-2}$

V. Conclusion

This paper presented a novel Neyman-Pearson approach for Compressed Sensing Multiuser Detection by utilizing activity Log-Likelihood ratios. This approach allows to tightly control activity error rates as missed detection or false alarm. In particular, our algorithm allows detection at a constant false alarm or missed detection rate while the counterpart is as small as possible. This approach gives nearly full control over activity errors and thereby simplifies eases to meet system requirements given from higher layers.

Acknowledgment

This work was founded by the German Research Foundation (DFG) under grant DE 759/3-1. Part of this work has been performed in the framework of the FP7 project ICT-317669 METIS, which is partly funded by the European Union. The authors would like to acknowledge the contributions of their colleagues in METIS, although the views expressed are those of the authors and do not necessarily represent the project.

Appendix A
Proof of Theorem 1

The sake of (11) is to provide estimates for the false alarm and missed detection probabilities. We first rewrite the missed detection and false alarm probabilities in terms of activity LLRs L_k for node k. The decision regions R_A and R_I are fully determined by the activity LLRs as $R_I = \{L_k : L_k \geq t\}$ and $R_A = \{L_k : L_k < t\}$. This allows us to write missed detection and false alarm probability for node k as

$$\mathrm{Pr}_{\mathrm{Md}}(t) = \mathrm{Pr}(x_k \in \mathcal{A}) \int_{R_I(t)} p(L_k | x_k \in \mathcal{A})\, dL_k \tag{14a}$$

$$\mathrm{Pr}_{\mathrm{Fa}}(t) = \mathrm{Pr}(x_k = 0) \int_{R_A(t)} p(L_k | x_k = 0)\, dL_k. \tag{14b}$$

In the following we only focus on false alarm probabilities as the proof for the missed detection probability is nearly the same. The proof outlined here bases on the assumption that the error due to the approximation of the activity LLRs with the max-log approximation is negligible. The posteriori probability obtained from the activity LLRs $\mathrm{Pr}(x_k = 0|L_k)$ is a sample of a random variable with support $L_k \in \mathbb{R}$. This is due to the fact that the activity LLRs are random variables. In the following we show that the set specific averages of this random variable converge to (14b). We therefore rewrite (14b) with the application of the Bayes-Rule as

$$\mathrm{Pr}_{\mathrm{Fa}}(t) = \int_{R_A(t)} \mathrm{Pr}(x_k = 0|L_k)\, p(L_k)\, dL_k \tag{15}$$

We see that (15) is the mean of the random variable $\mathrm{Pr}(x_k = 0|L_k)$ over the interval $R_A(t)$. We now assume that $|\mathcal{L}_A| \to \infty$. $\mathrm{Pr}(x = 0|L_k) = \frac{1}{1+\exp(-L_k)}$ allows us to sample this random variable, where each calculated activity LLR generates one sample. Averaging these samples gives the average missed detection probability $\tilde{\mathrm{Pr}}_{\mathrm{Fa}}$ and we have

$$\tilde{\mathrm{Pr}}_{\mathrm{Fa}}(\mathcal{L}, t) = \lim_{|\mathcal{L}_A| \to \infty} \frac{\sum_{k \in \mathcal{L}_A} \mathrm{Pr}(x_k = 0|L_k)}{|\mathcal{L}_A|} \to \int_{R_A(t)} \mathrm{Pr}(x_k = 0|L_k)\, p(L_k)\, dL_k. \tag{16}$$

With the law of large numbers, the sample mean $\tilde{\mathrm{Pr}}_{\mathrm{Fa}}(\mathcal{L}, t)$ converges to the true mean $\mathrm{Pr}_{\mathrm{Fa}}(t)$ if the size of the set $|\mathcal{L}_A|$ is sufficiently large. □

A. Convergence

In the following we define the approximation error as $\Delta = |\tilde{\mathrm{Pr}}_{\mathrm{Fa}}(\mathcal{L}, t) - \mathrm{Pr}_{\mathrm{Fa}}(t)|$. The Chebyshev inequality [18] allows to bound the error Δ via

$$\mathrm{Pr}\{\Delta \geq \epsilon\} \leq \frac{\sigma^2_{\tilde{\mathrm{Pr}}_{\mathrm{Fa}}}}{\epsilon^2}, \tag{17}$$

with $\sigma^2_{\tilde{\mathrm{Pr}}_{\mathrm{Fa}}}$ being the variance of the estimate $\tilde{\mathrm{Pr}}_{\mathrm{Fa}}(\mathcal{L}, t)$. In the following we consider the variance of $\tilde{\mathrm{Pr}}_{\mathrm{Fa}}(\mathcal{L}, t)$ for which we assume the activity LLRs to be i.i.d. In this case, the variance can be calculated via

$$\begin{aligned} \sigma^2_{\tilde{\mathrm{Pr}}_{\mathrm{Fa}}} &= \mathrm{var}\left[\frac{\sum_{k \in \mathcal{L}_A} \mathrm{Pr}(x = 0|L_k)}{|\mathcal{L}_A|}\right] \\ &= \frac{1}{|\mathcal{L}_A|} \mathrm{var}(\mathrm{Pr}(x = 0|L_k)). \end{aligned} \tag{18}$$

To obtain $\mathrm{var}(\mathrm{Pr}(x = 0|L_k))$, we use a first order Taylor expansion and obtain $\frac{d}{dL_k} \frac{1}{1+\exp(-L_k)} = \frac{\exp(L_k)}{(1+\exp(L_k))^2}$. The variance can now be approximated as [18]

$$\mathrm{var}(\mathrm{Pr}(x = 0|L_k)) \approx \left[\frac{\exp(\bar{L}_k)}{(1+\exp(\bar{L}_k))^2}\right]^2 \sigma^2_{L_k}, \tag{19}$$

where $\sigma^2_{L_k}$ denotes the variance and $\bar{L}_k$ denotes the mean of the activity LLR, respectively. The following steps assume that the approximation (19) is sufficiently tight. Under this condition, we obtain an upper bound for the approximation error after inserting (19) into (18). Together with (17) we have

$$\Pr\{\Delta \geq \epsilon\} \leq \left[\frac{\exp\left(\bar{L_k}\right)}{(1+\exp(\bar{L_k}))^2}\right]^2 \frac{\sigma^2_{L_k}}{|\mathcal{L}_A|\epsilon^2}. \tag{20}$$

Considering (20) shows that the approximation error depends on several parameters. First, we see that making $|\mathcal{L}_A|$ large decreases the approximation error down to arbitrary small values. This reveals that the cardinality of the sets $|\mathcal{L}_A|$ and $|\mathcal{L}_I|$ is crucial for the performance of the Neyman-Pearson detection. Another side effect is that the mean of the activity LLR $\bar{L}_k$. If the magnitude of $\bar{L}_k$ is large, the approximation error is negligible as the first factor of (20) converges to zero. This also shows that activity LLRs with mean value close to zero lead to high approximation errors.

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