Spectral Shaping for Faster-Than-Nyquist Signaling

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***Abstract*—Spectral shaping is applied to Faster-Than-Nyquist (FTN) signaling to improve information rates. Under a transmit power constraint, the best shape is an inverse filter but for root raised cosine pulses this requires infinite power at the shaper input. Both input and output power constraints at the shaper are thus proposed as a solution to maximize the information rate while still guaranteeing practicality.**

I. Introduction

Mazo [1] introduced Faster-Than-Nyquist (FTN) signaling for sinc pulses that are modulated faster than the Nyquist rate. He proved that sending sinc pulses 25% faster than this rate does not decrease the minimum Euclidean distance between sequences of binary modulated symbols. The resulting inter-symbol interference (ISI) introduced at the transmitter must be eliminated at the receiver.

Sinc pulses can not be realized due to their infinite impulse response. More importantly, they decay very slowly in time. Therefore other pulses are used, for example raised cosine (RC) or root raised cosine (RRC) pulses. Rusek and Anderson [2], [3] investigated such pulses when the input is constrained to be a sequence of Gaussian independent and identically distributed (i.i.d.) symbols. They showed that FTN achieves higher information rates than Nyquist ISI-criterion signaling due to the *excess* pulse bandwidth, which is the extra bandwidth of a pulse compared to the bandwidth of a sinc pulse.

Recent studies propose to use spectral shaping with FTN signaling to maximize the mutual information rate. In [2], it was mentioned that spectral shaping can increase information rates, and in [4] classical waterfilling was applied to cyclostationary FTN signaling with receiver mismatch. In [5], waterfilling algorithms with shortened channel detectors were used to derive optimal transmit filters. We also study spectral shaping for FTN. The natural approach is to use orthogonal frequency division multiplexing (OFDM) where a band $[-A/2, A/2]$ Hz is divided into subcarriers f_k, $1 \leqslant k \leqslant K$, having frequency bins of width $\Delta f = A/K$. Fig. 1 shows an OFDM transmit system having an RRC transmit filter. U is the vector of random bits that pass through a Gaussian modulator which pass K symbols to the inverse fast Fourier transform (IFFT) block where the spectral shaping takes place. X is the input vector to the transmit filter, Z is the white Gaussian noise and Y is the output vector.

The subcarriers are centered at $f_k = \left(k - \frac{K+1}{2}\right)\Delta f$ and the FTN capacity is (see [2])

$$C_{\text{FTN}} = \sum_{k=1}^{K} \log_2\left(1 + \frac{P|H(f_k)|^2}{N_0}\right)\Delta f \qquad (1)$$

where P is the average transmit power, N_0 is the noise power spectral density, and $H(f)$ is the spectrum of the pulse $h(t)$ having unit energy. To avoid aliasing, the FTN factor τ should satisfy $0 < \tau \leqslant (AT)^{-1}$ [2]. The FTN capacity and the Nyquist/orthogonal transmission capacity are evaluated in [6] showing that at high SNR the asymptotic gain for FTN signaling is equal to the *excess* bandwidth. With spectral sharing, however, the most spectrally efficient pulses are Shannon's sinc pulses (see [7]).

In this paper we discuss a particular requirement for RRC pulses, namely that one must put a power constraint *before* the transmit filter and not only after the transmit filter. The paper is organized as follows, Section II reviews waterfilling and Section III defines a new approach to spectral shaping that combines waterfilling with inverse filtering. Section IV discusses the results of the proposed solution.

II. Preliminaries

This section reviews classic waterfilling. The objective is to maximize the information rate by distributing power over the channel bandwidth. We consider a dispersive channel model [8], [9], [10] where we interpret a transmit filter using FTN signaling as the channel. Consider the input vector $\mathbf{X} = (X_1, X_2, ..., X_K)$ and the output vector $\mathbf{Y} = (Y_1, Y_2, ..., Y_K)$. We have

$$Y_k = H(f_k)X_k + Z_k \qquad (2)$$

where $H(f)$ is the transfer function of the channel, $1 \leqslant k \leqslant K$, and the X_k are zero-mean independent Gaussian inputs with powers $Q_k\Delta f$. The input power is constrained as

$$\sum_{k=1}^{K} Q_k\Delta f \leqslant Q. \qquad (3)$$

The $Z_k \sim \mathcal{CN}(0, N_k)$ are independent complex Gaussian noise symbols with $N_k = N_0\Delta f$ if the noise is white.

The mutual information is computed as [10]

$$I(\mathbf{X};\mathbf{Y}) = h(\mathbf{Y}) - h(\mathbf{Y}|\mathbf{X}) = h(\mathbf{Y}) - h(\mathbf{Z}) \qquad (4)$$

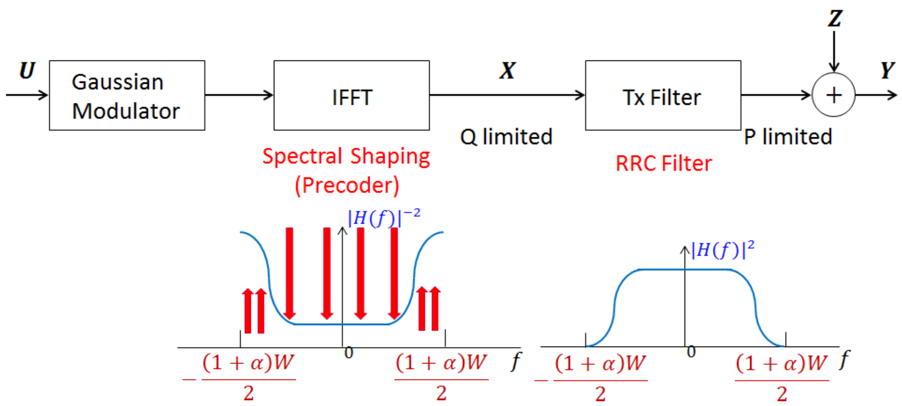

Fig. 1. OFDM transmit system with spectral shaping considering two power constraints.

where

$$h(\mathbf{Y}) = h(Y_1 Y_2 .... Y_K) \leqslant \sum_{k=1}^{K} h(Y_k) \tag{5}$$

and $h(Y_k)$ is the differential entropy which is defined as

$$h(Y_k) = -E[\log_2 p(Y_k)] = -\int p(y_k) \log_2 p(y_k)\, dy_k \tag{6}$$

and

$$h(\mathbf{Z}) = \sum_{k=1}^{K} h(Z_k) \tag{7}$$

due to the noise independence. Consequently, the mutual information is upper bounded as [9, p. 252]

$$\begin{aligned} I(\mathbf{X};\mathbf{Y}) &\leqslant \sum_{k=1}^{K} [h(Y_k) - h(Z_k)] \\ &\leqslant \sum_{k=1}^{K} \log_2 \left(1 + \frac{|H(f_k)|^2 Q_k}{N_0}\right) = C(\mathbf{Q}) \end{aligned} \tag{8}$$

where $\mathbf{Q} = (Q_1, Q_2, ..., Q_K)$. To maximize (8), consider the Lagrangian

$$J(\mathbf{Q}) = C(\mathbf{Q}) + \sigma \left(\frac{Q}{\Delta f} - \sum_{k=1}^{K} Q_k\right). \tag{9}$$

Setting the derivative with respect to Q_k to zero, we get

$$\sigma = \left(\frac{|H(f_k)|^2}{N_0 + |H(f_k)|^2 Q_k}\right) \log_2 e. \tag{10}$$

We also need to satisfy $Q_k \geqslant 0$. The waterfilling power allocation is thus

$$Q_k = \max\left(0, \frac{\log_2 e}{\sigma} - \frac{N_0}{|H(f_k)|^2}\right) \tag{11}$$

where σ is a constant chosen to satisfy the power constraint (3) with equality.

III. Spectral Shaping

The constraint (3) is not really required because the true transmit power constraint is after the pulse shaper rather than before it. The transmit power constraint is

$$\sum_{k=1}^{K} Q_k |H(f_k)|^2 \Delta f \leqslant P \tag{12}$$

where P is the power after the transmit filter. However, if we use only the constraint (12), then the best spectral shaping is an inverse filter that mimics a sinc pulse spectrum. For example, consider an RRC pulse as in Fig. 2 (a) with the spectrum

$$|H(f)|^2 = \begin{cases} 1/W, & |f| \leqslant (1-\alpha)\frac{W}{2} \\ \frac{1}{2W}\left[1 + \cos\left(\frac{\pi}{\alpha W}\left[|f| - (1-\alpha)\frac{W}{2}\right]\right)\right], & \\ \qquad (1-\alpha)\frac{W}{2} < |f| \leqslant (1+\alpha)\frac{W}{2} & \\ 0, & \text{else} \end{cases} \tag{13}$$

where $A = (1+\alpha)W$. We apply an inverse of the RRC pulse so that we put more power on the weak tones and less power on the strong tones, see Fig. 2 (b). The result is a sinc pulse of bandwidth $(1+\alpha)W$ as in Fig. 2 (c). But this requires a large power at the input of the transmit filter, as shown in Appendix A. Hence we need to consider the constraint (3).

The optimization problem that we consider is

$$\begin{aligned} \text{maximize} \quad & \sum_{k=1}^{K} \log_2 \left(1 + \frac{Q_k |H(f_k)|^2}{N_0}\right) \\ \text{subject to} \quad & \sum_{k=1}^{K} Q_k |H(f_k)|^2 \leqslant P/\Delta f \\ & Q_k \geqslant 0, \quad \forall k \\ & \sum_{k=1}^{K} Q_k \leqslant Q/\Delta f. \end{aligned} \tag{14}$$

This problem is a convex optimization problem, and its solution gives (see Appendix B):

$$Q_k = \max\left(0, \frac{\log_2 e}{\delta + \gamma |H(f_k)|^2} - \frac{N_0}{|H(f_k)|^2}\right). \tag{15}$$

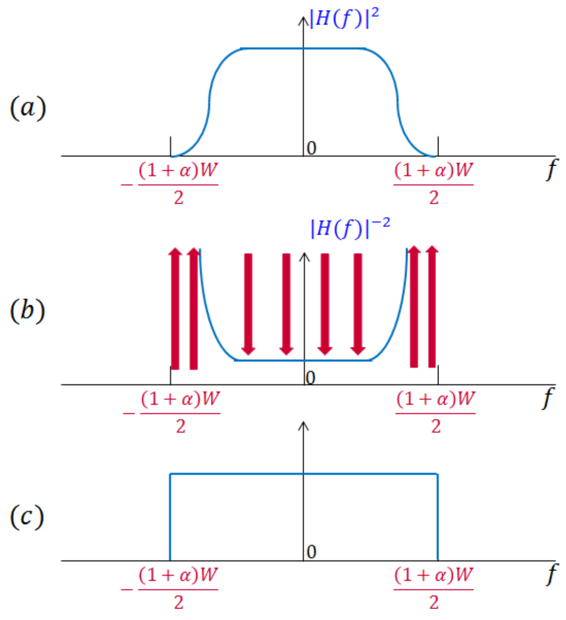

Fig. 2. Inverse filtering applied on FTN using RRC pulse.

We may use convex optimization routines to solve the problem. To get more insight, consider the following cases. First, suppose the optimizing Q_k do not meet the constraint (3) with equality. We then have $\delta = 0$ and

1) The variances Q_k are

$$Q_k = \max\left(0, \left(\frac{\log_2 e}{\gamma} - N_0\right)\frac{1}{|H(f_k)|^2}\right). \quad (16)$$

2) The constraint (12) is met with equality.

Equation (16) describes an inverse filter for that part of the band in which the transmitter is active. Thus, we mimic a sinc pulse of bandwidth at most $(1+\alpha)W$.

Second, suppose the optimizing Q_k do not meet (12) with equality. We then have $\gamma = 0$ and

1) The variances Q_k are

$$Q_k = \max\left(0, \frac{\log_2 e}{\delta} - \frac{N_0}{|H(f_k)|^2}\right). \quad (17)$$

2) The constraint (3) is met with equality.

Equation (17) describes a classic waterfilling filter (see (11)). Thus, if only one of the constraints is tight, either waterfilling or inverse filtering is the optimal spectral shaping. Note that the maximum Q so that waterfilling is best depends on the SNR and on K. In particular, if $K \to \infty$ then inverse filtering requires infinite energy, and thus a finite Q implies that a waterfilling filter will be used.

IV. Results and Discussions

We compare the information rates of sinc pulses, an RRC pulse without spectral shaping and an RRC pulse with spectral shaping having the two power constraints. Fig. 3 shows the spectral efficiency curves for an RRC pulse with a roll-off factor $\alpha = 0.5$, having system parameters $\tau = 2/3, T = 1/2, A = 3$, and $K = 1024$, with spectral shaping with two power constraints where Q is limited to $10^2, 10^3, 10^4$. Observe that as Q increases the information rate increases so that for very large Q, the RRC curve coincides with the AWGN channel curve.

V. Conclusion

Spectral shaping with a power constraint after the transmit filter achieves the highest information rate but requires infinte power at the filter input. The solution we propose is to apply spectral shaping with power constraints before and after the transmit filter.

Appendix A

Consider an RRC filter for which the inverse filter has the form

$$|H(f)|^{-2} = c(1+\cos(af-b))^{-1} \quad (18)$$

for $|f| \in \left[\frac{b}{a}, \frac{b+\pi}{a}\right]$ where a, b, c are positive constants. We have

$$\begin{aligned}\int_{\frac{b}{a}}^{\frac{b+\pi}{a}} |H(f)|^{-2}\, df &= \int_0^{\pi} \frac{c}{a}(1+\cos(f))^{-1}\, df \\ &= \frac{c}{a}\, \frac{\sin(f)}{1+\cos(f)}\Big|_{f=0}^{\pi} \\ &= \frac{c}{a} \lim_{f\to\pi^-} \frac{\sin(f)}{(1+\cos(f))}\end{aligned}$$

where $\lim_{f\to\pi^-}$ means we approach π from below. Using l'Hôpital's rule, we get

$$\lim_{f\to\pi^-} \frac{\sin(f)}{(1+\cos(f))} = \lim_{f\to\pi^-}\left(-\frac{1}{\tan(f)}\right) = \infty.$$

This shows that we need infinite energy/power at the input of the transmit filter.

Appendix B

Consider the optimization problem (14). The Lagrangian is [11]:

$$\sum_{k=1}^{K} \log_2\left(1 + \frac{Q_k|H(f_k)|^2}{N_0}\right) + \gamma\left(\frac{P}{\Delta f} - \sum_{k=1}^{K} Q_k|H(f_k)|^2\right) + \left(\sum_{k=1}^{K} \lambda_k Q_k\right) + \delta\left(\frac{Q}{\Delta f} - \sum_{k=1}^{K} Q_k\right) \quad (19)$$

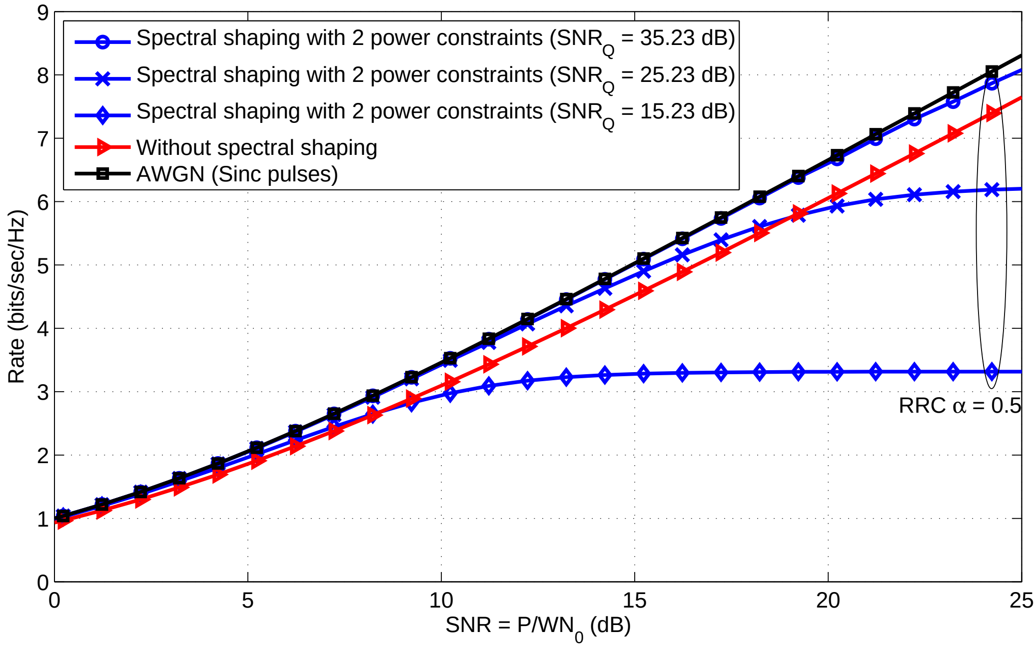

Fig. 3. Information rates for sinc pulses and RRC pulses with $\alpha = 0.5$ and $Q = 10^2, 10^3, 10^4$.

The KKT conditions are:

$$Q_k \geqslant 0, \lambda_k \geqslant 0, \lambda_k Q_k = 0 \tag{20}$$

$$\sum_{k=1}^{K} Q_k |H(f_k)|^2 \leqslant P/\Delta f, \gamma \geqslant 0 \tag{21}$$

$$\gamma \left(\frac{P}{\Delta f} - \sum_{k=1}^{K} Q_k |H(f_k)|^2 \right) = 0 \tag{22}$$

$$\sum_{k=1}^{K} Q_k \leqslant Q/\Delta f, \delta \geqslant 0 \tag{23}$$

$$\delta \left(\frac{Q}{\Delta f} - \sum_{k=1}^{K} Q_k \right) = 0 \tag{24}$$

The derivative of (19) with respect to Q_k should be 0, which gives

$$\lambda_k = \gamma |H(f_k)|^2 + \delta - \frac{|H(f_k)|^2 \log_2 e}{N_0 + Q_k |H(f_k)|^2} \tag{25}$$

The KKT condition (20) implies that $Q_k = 0$ or $\lambda_k = 0$. Thus using (23) we have

$$Q_k = \max \left(0, \frac{\log_2 e}{\delta + \gamma |H(f_k)|^2} - \frac{N_0}{|H(f_k)|^2} \right). \tag{26}$$

The KKT conditions (21) and (23) give

$$\sum_{k:Q_k>0} \left(\frac{\log_2 e}{\delta + \gamma |H(f_k)|^2} - \frac{N_0}{|H(f_k)|^2} \right) |H(f_k)|^2 \leqslant \frac{P}{\Delta f} \tag{27}$$

and

$$\sum_{k:Q_k>0} \left(\frac{\log_2 e}{\delta + \gamma |H(f_k)|^2} - \frac{N_0}{|H(f_k)|^2} \right) \leqslant \frac{Q}{\Delta f}. \tag{28}$$

From the objective, we see that if both (3) and (12) are loose then we may increase some Q_k. From (26), we see that this can be accomplished by decreasing δ or γ. Moreover, the optimal solution will satisfy one of three conditions: (1) $\delta = 0, \gamma > 0$; (2) $\delta > 0, \gamma = 0$; (3) $\delta > 0, \gamma > 0$. Case (1) will happen for small P or large Q, since then (27) will be tight, (28) will be loose, but the complementary slackness conditions (22) and (24) are both satisfied. Similarly, case (2) will happen for large P or small Q. Case (3) is the intermediate case where both (27) and (28) will be tight.

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