Energy-Aware Activation of Nomadic Relays for Performance Enhancement in Cellular Networks

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ABSTRACT—This paper presents an optimization framework for energy-aware relay selection and user association in cellular networks aided by nomadic relays. The framework of sparse optimization is used to minimize network energy consumption for desired service provisioning of the terminals. We show some of the optimization constraints are of quadratic form due to the assumption of relays with wireless backhaul links. Hence, previously proposed algorithms for activation of network elements with wired backhaul links are not applicable. In this paper, therefore, novel algorithms based on different relaxations of the quadratic constraints are proposed and evaluated for energy savings. Simulation results confirm that the proposed algorithms may significantly reduce the overall energy consumption of cellular networks when compared with other activation algorithms.

I. INTRODUCTION

The relaying concept has proven to be a cost-efficient solution to accommodate the increasing demand on capacity and coverage for wireless data access [1]. In the past decade, a tremendous effort has been expended by standardization groups, aiming at the smooth roll-out and industrialization of the relaying concept [2], [3]. Since the souring price for Operations & Maintenance (O&M) prevents the operators from a massive relay node (RN) deployment, opportunistic approaches using user equipment (UE)s as RNs have attracted attention from academia to industry [4].

A 5G system component that tries to respond to the increasing traffic volume of the future information society is the concept of a nomadic network [5], which is defined as a network with randomly distributed nodes (e.g., parked vehicles with on-board relay infrastructure and advanced backhaul antennas) acting as potential relays between UEs and base station (BS)s. While the location of operator-deployed relay nodes is optimized by means of network planning, the RNs in a nomadic network are randomly distributed. Moreover, their availability and position may change in time (hence, the term “nomadic”). The nomadic relay nodes operate in a self-organized fashion and are activated and deactivated based on capacity, coverage, load balancing or energy efficiency demands. Therefore, the concept of a nomadic network describes an effective extension of the cellular infrastructure.

Assuming that a large number of nomadic RNs is available (e.g., parked vehicles in an urban scenario), there is much room for optimization with respect to RN selection as well as the UE and BS assignment to the selected RNs. Furthermore, radio resources must be flexibly assigned among BSs, RNs and UEs, to fully exploit the relaying gains and fulfill the Quality of Service (QoS) requirements of all the UEs. In summary, important aspects in such networks include:

- Relay Selection (RS): selection and subsequent assignment of RNs to BSs;
- User Association (UA): assignment of UEs to RNs (two-hop relaying) or BSs (direct communications);
- Radio Resource Management (RRM): a flexible allocation of resources available at BSs to RNs and UEs.

Note that for the maximum spectral efficiency, RNs are assumed to reuse the resources of BSs.

Regarding the related work, the authors in [6], [7], provide a general overview of relay networks and address important deployment aspects. Recently, the concept of self-organizing relays has been discussed in [8], where the authors propose a dynamic resource sharing strategy taking into account the varying UE data traffics, however, the RS and UA problems are not addressed. The topic of RS has been very well elaborated theoretically in [9], [10], where the optimization approach is based on link level metrics such as outage performance and QoS provisioning, rather than a network-wide performance enhancement. The work in [11] proposes a network load balancing optimization algorithm for RS, but without incorporating reuse and flexible resource sharing among different network elements. In [12], a joint UA and flexible link scheduling algorithm is proposed in order to maximize the network throughput in single-cell relay assisted networks.

In this paper, we propose an optimization framework for the aforementioned RS and UA problems in nomadic relay-assisted multi-cell networks, where flexible resource sharing among all the nodes is considered. With the goal of enhancing the energy efficiency in mind, we formulate an optimization problem that takes into account wireless backhaul links for the nomadic RNs. We show that the wireless backhaul links for the RNs results in optimization constraints of quadratic form. Therefore, sparse optimization methods, as proposed in [13] for cellular networks with wired backhaul links, cannot be applied directly. We propose various algorithmic solutions.
by combining the approach of [13] with some relaxation methods for quadratically constrained problems [14]. Finally the algorithms are numerically evaluated in terms of the overall energy savings.

The rest of the paper is organized as follows: Section II describes the system model and the optimization framework, while Section III presents algorithms for minimizing the energy consumption. Simulation results are given in Section IV that is followed by some concluding remarks.

II. SYSTEM MODEL AND PROBLEM STATEMENT

In this work, we consider the downlink channel of a nomadic relay network with $M$ BSs, $N$ UEs and $K$ RNs. The set of BSs, UEs and RNs are denoted by, respectively, $B$, $U$ and $R$. Furthermore, we use direct links, access links and relay links to denote the BS-UE, RN-UE and BS-RN links, respectively.\(^2\) In this paper, we consider L3-RNs according to the Long Term Evolution (LTE) standard [2]. Such RNs have all the RRM functionalities of the LTE BSs: the RNs are able to reuse the resources of the BS for the access link transmissions, while the relay links and direct links compete for resources allocated to the corresponding BS.

The bandwidths available at BSs and RNs are fixed and grouped in vectors $b^{(m)} = (b_1^{(m)}, \ldots, b_K^{(m)})^T$ and $b^{(k)} = (b_1^{(k)}, \ldots, b_K^{(k)})^T$, respectively. The vector of required minimum rates of the UEs is denoted by $r^{(n)} = (r_1^{(n)}, \ldots, r_N^{(n)})^T$. Note that each UE can be connected either to an RN or directly to a BS, while an RN can be connected to a BS or detached from the network. The Spectral Efficiency (SE) of a connection from any node $i$ to any node $j$ is assumed to be

$$\omega_{i,j} = \zeta_b \cdot \log(1 + \zeta_s^{-1} \cdot \tau_{i,j}).$$

(1)

Here and hereafter, $\zeta_b > 0$ is the bandwidth efficiency and $\zeta_s > 0$ is used to denote the Signal to Interference plus Noise Ratio (SINR) efficiency, while $\tau_{i,j}$ denotes the corresponding SINR. Throughout the paper, (A.1) $r^{(n)}$, $b^{(m)}$, $b^{(k)}$ and $\omega_{i,j}$ are known parameters or can be estimated reliably.

To ensure coverage when network elements are deactivated for energy savings, we follow the approach of [13] and assume the worst-case interference scenario in the downlink. Moreover, we make the following assumption:

(A.2) While access links and direct links interfere with each other, the RNs use separate time-frequency resources on the relay links and access links. Hence the access links do not interfere with the relay links.

So, the SINR of a direct/access link $(i,j)$ at UE $j$ is given by

$$r_{i,j}^{(n)} = \frac{P_{i,j}}{\sum_{d \in R, d \neq i} P_{d,j} + N_j}.$$

(2)

where $N_j$ is the receiver-side noise power and $P_{d,j}$ is the received signal or interference power at node $j$, either from a base station $d \in B$ or from a relay station $d \in R$. Similarly, the SINR of a relay link $(i,j)$ at RN $j$ yields

$$r_{i,j}^{(k)} = \frac{P_{i,j}}{\sum_{d \in B, d \neq i} P_{d,j} + N_j}.$$

(3)

where, due to Assumption (A.2), base stations in $B$ are the only interference sources.

Remark 1. Note that the worst-case interference can be very severe in dense network. In such case, the presented approach, which is based on the worst-case interference, may be highly suboptimal. Therefore, we implicitly assume that other mechanisms are implemented such as clustering or scheduling to avoid strong interference between very closely located network elements.

A. Capacity constraints

The required minimum rates impose together with the limited bandwidth at nodes (BS or RN) $M + K$ constraints to the system. In other words, the available bandwidth $b^{(m)}$ at a BS must be sufficient to support the required rates at the assigned relay links and direct links, whereas the bandwidth demand on the access links of a RN may not exceed $b^{(k)}$.

Hence, for BS $i \in B$, we have

$$\sum_{j \in U} \omega_{i,j} x_{i,j} + \sum_{j \in R} \omega_{i,j} x_{i,j} \leq b^{(m)}, i \in B$$

(4)

while the constraint associated with RN $i \in R$ can be written as

$$\sum_{j \in U, \omega_{i,j}} x_{i,j} \leq b^{(k)}, i \in R.$$

(5)

Here and hereafter, $x_{i,j}$ is the assignment variable: $x_{i,j} = 1$ if there is an active connection between node $i$ and node $j$, and $x_{i,j} = 0$ otherwise. Notice that $r^{(k)}_j$ is the rate requirement of an RN which is the sum rate of all the UEs connected to it:

$$r^{(k)}_j = \sum_{k \in U} r^{(n)}_{j,k}.$$

(6)

Notice that if the constraint is satisfied, then the required minimum rates are met for each user and the available bandwidths are not exceeded.

Before formulating the problem, let us write the constraints in a more compact matrix form. To this end, we introduce the assignment matrix to be defined as

$$X \triangleq \begin{pmatrix} X^{(m,n)} \\ X^{(k,n)} \end{pmatrix} \in \{0,1\}^{(M+K) \times (N+K)}$$

(7)

where $0^{K \times K}$ is an all zero matrix of size $K \times K$, while $X^{(m,n)} \in \{0,1\}^{M \times N}$, $X^{(k,n)} \in \{0,1\}^{K \times N}$ and $X^{(m,k)} \in \{0,1\}^{M \times K}$ are assignment matrices for the direct, access

\(^2\)Throughout this paper, notations with superscripts (m), (n) and (k) are variables associated with BSs, UEs and RNs, respectively, while notations with (m,n), (k,n) and (m,k) are referring to the direct links, access links and relay links, respectively.

\(^3\)Throughout the paper, $1^I(0^I)$ refers to column vector of length $I$. If not specified, $1^I \in \mathbb{R}^I$ is a column vector with proper length for matrix operator. Furthermore, $1^m \times n(0^m \times n)$ refers to an $m \times n$ matrix of ones/zeroes.
and relay links, respectively. Furthermore, let \( \Omega^{(m,n)} := (\frac{1}{c_{i,j}(m,n)})_{i,j} \in \mathbb{R}^{M \times N} \), \( \Omega^{(m,k)} := (\frac{1}{c_{i,j}(m,k)})_{i,j} \in \mathbb{R}^{K \times N} \) and \( \Omega^{(k,a)} := (\frac{1}{c_{i,j}(k,a)})_{i,j} \in \mathbb{R}^{M \times K} \). Further, let \( W = (W^{(m,n)}W^{(m,k)}W^{(k,a)})_{0 \times K+N} \) where the non-zero blocks are defined to be

\[
W^{(m,n)} = (1_{M}r^{(m,n)}T) \circ \Omega^{(m,n)} \quad (8a)
\]

\[
W^{(k,a)} = (1_{K}r^{(k,a)}T) \circ \Omega^{(k,a)} \quad (8b)
\]

\[
W^{(m,k)} = (1_{M}r^{(m,k)}T) \circ \Omega^{(m,k)} \quad (8c)
\]

Here \( r^{(k)} \) is the vector of RN rate requirements \( (r^{(k)}) \):

\[
r^{(k)} = X^{(k,a)} \cdot r^{(n)} = (r_{1}^{(k)}, \ldots, r_{K}^{(k)})^{T}. \quad (9)
\]

With these definitions in hand, it may be easily verified that the constraints in (4) and (5) can be written as

\[
(W \circ X) \cdot 1 \leq 1.
\]

### B. Problem Formulation

Now we are in a position to formulate our optimization problem. The goal is to find an assignment of UEs to RNs and RNs to BSs so as to minimize a predefined cost function \( U(X) : \{0, 1\}^{M+K \times N+K} \rightarrow \mathbb{R} \) subject to the capacity and connectivity constraints defined in the previous section:

\[
\begin{align*}
\text{min} & \quad U(X) := \sum_{i=1}^{M+K} c_{i}||e_{i}^{T}X_{1}||_{0} \quad (10a) \\
\text{subject to} & \quad X^{T} \cdot 1 = 1 \quad (10b) \\
& \quad (W \circ X) \cdot 1 \leq 1 \quad (10c) \\
& \quad X \in \{0, 1\}^{(M+K) \times (N+K)}. \quad (10d)
\end{align*}
\]

While (10c) captures the capacity constraints, (10b) results from the demand on network connectivity: every UE and RN must be connected to a network. In (10a), the vector \( c = (c_{1}, \ldots, c_{M+K}) \in \mathbb{R}^{M+K+1} \) reflects the energy consumption of active BSs and RNs, while \( e_{i}^{T} \) is the \( i \)-th row of the identity matrix of the same size as \( X \), so that \( e_{i}^{T}X \) yields the \( i \)-th row of \( X \). Finally, for any vector \( x \), its \( l_{0} \)-norm is denoted by \( ||x||_{0} \), which is the cardinality of the set of non-zero vector elements of \( x \).

Therefore, given \( X \), the cost function \( U(X) \) captures to the overall energy consumed of all active nodes (network elements). From this, we can conclude that minimizing the cost function will in general lead to significant energy savings when the optimal vector is sparse. Note that by (8) and (9), \( W^{(m,n)} \) and \( W^{(k,a)} \) are constant matrices, while \( W^{(m,k)} \) contains \( X^{(k,a)} \). By (8), (9) and (10c), it implies that, whereas the presence of direct and access links leads to linear constraints, the constraints associated with the relay links are of quadratic form containing the cross product of \( X^{(k,a)} \) and \( X^{(m,k)} \). This stands in clear contrast to [13] where all the constraints are of linear form.

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4For any two matrices \( A \) and \( B \) of the same size, \( A \circ B \) denotes the Hadamard matrix product, while both \( A \leq B \) and \( A \geq B \) should be understood as element-wise comparisons.

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### III. Algorithms for Energy Saving

Problem (10) is notoriously difficult to solve. Its intractability can be attributed to the following two facts:

1) The objective function (10a) is a discrete metric defined on a discrete set (see (10d)).

2) The constraint (10c) contains product terms induced by the original variables, and hence the problem is a quadratically constrained problem [14].

The first fact implies that the problem is of combinatorial nature. Although a global optimum is out of reach, we can proceed essentially as in [13] to find almost optimal solutions by resorting to the Majorization Minimization (MM) techniques. First the problem in (10) is relaxed by considering an approximation of the \( l_{0} \)-norm in (10a) defined to be

\[
\tilde{U}_{\epsilon}(X) := \sum_{i=1}^{M+K} c_{i}(\log(\epsilon + e_{i}^{T}X_{1}) - \log(\epsilon)) \quad (11)
\]

for some fixed \( \epsilon > 0 \) that is sufficiently small. The function in (11) can be written in a more compact form [13] that reveals the possibility of minimizing (11) by solving a sequence of linear programs:

\[
X^{(l+1)} \in \text{argmin}_{X} \sum_{i=1}^{M+K} c_{i}e_{i}^{T}X_{1} \quad (12)
\]

With the discrete constraint \( X \in \{0, 1\}^{(M+K) \times (N+K)} \) being relaxed to a continuous one \( X \in [0, 1]^{(M+K) \times (N+K)} \) we finally arrive at a sequence of convex optimization problems with the \( l \)-th iteration given by

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{M+K} c_{i}e_{i}^{T}X_{1} \\
\text{subject to} & \quad X^{T} \cdot 1 = 1 \quad (13a) \\
& \quad (W \circ X) \cdot 1 \leq 1 \quad (13b) \\
& \quad X \in [0, 1]. \quad (13c)
\end{align*}
\]

Since not all the constraints are linear, we have to extend the approach of [13] to deal with quadratic constraints. In the following, we present different approaches to Problem (13).

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### A. Iterative Backhaul Update

As the quadratic constraint arises from the inner product \( W^{(m,k)} \circ X^{(m,k)} \), where \( W^{(m,k)} \) depends on \( X^{(k,a)} \), we can conclude that the constraint becomes linear for any fixed \( X^{(k,a)} \). This suggests a very simple heuristic approach in which \( W^{(m,k)} \) defined by (8c) is iteratively updated, with the term \( X^{(k,a)} \) being kept fixed at a value obtained from the previous iteration. The steps of the algorithm are summarized in Algorithm 1.

6The algorithm is started with an initial value \( X^{(k,a)}(0) = 0 \) where the zero indicates that none of the UEs are assigned to the RNs. In particular, we may have \( (W^{(m,k)})(0) = 0 \) by (8c) and (9) . The problem in (13) is

\[
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\[
\|A\|_{F} = \sqrt{\sum_{i} \sum_{j} |a_{i,j}|^{2}} \text{ denotes the Frobenius norm of } A = (a_{i,j}).
\]

---

5Note that \( \lim_{k \rightarrow 0} \tilde{U}_{\epsilon}(X) = \tilde{U}_{\epsilon}(X) \).
Then, we can show that Problem (13) can be equivalently to a non-convex set, and 2) in a relaxation of this non-convex are the vectorization of $X$ and $x$ term by a new variable of the form quadratically constrained Problems (14). The key ingredient Linearization Technique (RLT) and apply them to the Semi-Definite Programming (SDP) and Reformulation-Relaxation methods to quadratically constrained problems are presented. First we need to write the optimization problem (14) in a standard form. To this end, we define $x = \text{vec}(X) := (x_1, \ldots, x_L)^T$, where $L = (M + K) \cdot (N + K)$ and $x_{ij} = \tilde{x}_{(M+K)(i-1)+j}$. Similarly, $x^{(m,n)}$, $x^{(k,n)}$ and $x^{(m,k)}$ are the vectorization of $X^{(m,n)}$, $X^{(k,n)}$ and $X^{(m,k)}$, respectively. Then, we can show that Problem (13) can be equivalently written as (for detailed derivation and explanation of $a_0$, $p_i$, $q_i$, and $Q_i$, the reader is referred to Appendix A)

\begin{align}
\min_{x} & \quad a_0^T x \tag{14a} \\
\text{subject to} & \quad p_i^T x = 0, \quad i \in \{1, \ldots, N + K\} \tag{14b} \\
& \quad \frac{1}{2} x^T Q_i x + q_i^T x \leq 1, \quad i \in \{1, \ldots, M + K\} \tag{14c} \\
& \quad 0 \leq x \leq 1. \tag{14d}
\end{align}

Unfortunately, the matrices $Q_i$ are in general indefinite so that the optimization problem is in general a non-convex Nonlinear Program (NLP). Since finding a global optimum is difficult, we resort in this paper to convex relaxation methods based on Linearization Technique (RLT) [16] and Reformulation-Linearization Technique (RLT) [16] and apply them to the quadratically constrained Problems (14). The key ingredient of the approaches lies in 1) the replacement of each product term by a new variable of the form $X := xx^T$ constrained to a non-convex set, and 2) in a relaxation of this non-convex constraint to a convex one.

In the case of a SDP relaxation, $X = xx^T$ is relaxed to $X = xx^T \succeq 0$ or, equivalently, $X := (\frac{1}{x} xx^T) \succeq 0$. ? An immediate consequence is that the SDP-based relaxation of (13) takes the form

\begin{align}
\min_{X} & \quad \text{tr}(A_0 X) \tag{15a} \\
\text{subject to} & \quad \text{tr}(P_i X) = 0, \quad i \in \{1, \ldots, N + K\} \tag{15b} \\
& \quad \text{tr}(Q_i X) \leq 0, \quad i \in \{1, \ldots, M + K\} \tag{15c} \\
& \quad 0 \leq x \leq 1 \tag{15d} \\
& \quad X \succeq 0. \tag{15e}
\end{align}

where $\text{tr}(\cdot)$ is used to denote the trace of a matrix and,

$A_0 = \begin{pmatrix} a^T_0 & 0_{L \times L} \\ 0_{L \times L} & 0_{L \times L} \end{pmatrix}$

$P_i = \begin{pmatrix} -2 & 0_{L \times L} \\ 0_{L \times L} & P_i \end{pmatrix}$

$Q_i = \begin{pmatrix} -2 & 0_{L \times L} \\ 0_{L \times L} & Q_i \end{pmatrix}$.

As pointed out in [14], semidefiniteness may remove large parts of the original feasible set, which in turn may lead to highly suboptimal solutions. The idea of [14] is to combine (or even substitute) the semidefiniteness constraint with some linear constraints obtained using RLT techniques in order to set some suitably chosen boundaries to the new variables. Proceeding essentially as in [14] shows that the additional constraints are

\begin{align}
\bar{X} \succeq 0_{L \times L}. \tag{16a} \\
X - 1x^T - x1^T \succeq -1_{L \times L}. \tag{16b} \\
X - x1^T \succeq 0_{L \times L}. \tag{16c}
\end{align}

The problem in (15) can be solved efficiently using some standard optimization tools, regardless of whether the linear constraints in (16) are considered or not.

IV. SIMULATION RESULTS

A. Scenarios

We consider a scenario including 7 LTE BSs with Inter Site Distance (ISD) of 500 m. To identify the key factors that significantly influence the performance of our proposed algorithms, we use different densities of RNs and vary rate requirements for UEs. The RNs are randomly distributed within the area covered by the BS. In order to avoid the very severe interference between cells that are located close to each other, we deactivate all RNs within a circle of 50 Meters of a BS. Furthermore for simplicity, any two RNs with a distance of less than 10 meters between them are assumed to use orthogonal resources (in time or frequency domain). An example of the deployment scenario and the simulation parameters are summarized in Tab. I.

7 $A \succeq 0$ means that matrix $A$ is semi-definite.
TABLE I
SIMULATION CONFIGURATIONS.

An example of the deployment scenario

Deployment scenario:
• 7 BSs in hexagon shape
• ISD: 500 m
• simulation area length: 1.2 km
• number of UEs: 40
• RN density: \{0, 10, 20, 30, 40\} / km²
• UE rate requirement: \{0.01, 0.1, 1, 10\} Mbps

Transmission Parameters
transmission power 46 dBm for BS & 23 dBm for RN
energy consumption 1000 Watt for BS & 20 Watt for RN
available bandwidth 10 MHz for BS & 10 MHz for RN
antenna configuration 2 antennas for BSs, RNs and UEs
noise figure 5 dB at UE & RN

Channel and Noise Parameters in [dB]
path loss model for all links as in Table A.2.1.1.2-3 in [17]
shadowing & fast fading not considered

Simulation Parameters
termination threshold \(\delta\) 0.00001
\(\epsilon\) for MM algorithm 0.0001

B. Results

We compare the following three algorithms:
• Closest Cell Selection (CCS): UEs and RNs are connected to the closest node,
• Iterative Backhaul Updating (IBU) as proposed in Section III-A,
• SDP and RLT Relaxation (SRR) as in III-B, where all the constraints in (15) and (16) are combined.

The performance on energy consumption of the three algorithms can be found in Fig. 1 and Fig. 2. It can be concluded from the figures that both IBU and SRR reduce the network energy consumption significantly. Furthermore, in this scenario, we observe that SRR outperforms IBU slightly, because IBU may, as discussed before, lead to local minimum. Figure 1 illustrates that the energy consumption decreases as the density of RNs increases; this is because in denser scenarios, there are more opportunities for relaying so that there is a higher probability for switching off BSs by redirecting attached UEs to RNs. From Fig. 2, we can see that the energy consumption increases as the UE rate requirements increase. If the UE rate requirement is very high, we note that RNs cannot significantly help to decrease the energy consumption, which can be attributed to the fact that antenna capabilities of RNs and UEs are assumed to be the same in the simulation, and therefore relaying with wireless backhaul links cannot offload the high demand on capacity.

V. CONCLUSION

In this paper, we presented a framework for relay selection and user association in nomadic relay networks, with the goal of achieving significant energy savings. Assuming relays with wireless backhaul links, we formulated an optimization problem using sparse optimization methods and showed that it is quadratically constrained. Moreover, we proposed heuristic algorithms based on different relaxations of the quadratic constraints. Numerical results are provided for an urban scenario showing that the proposed algorithms have considerable potential for energy savings in wireless networks with nomadic cells.

APPENDIX

A. Reformulation of the Problem

Since \(\bar{x} = \text{vec}(X)\), it can be easily verified that
\[
\begin{align*}
\mathbf{e}_i^T(\mathbf{X} \mathbf{1}) &= \text{vec}(\mathbf{1} \mathbf{e}_i^T \bar{x}) \\
\mathbf{e}_i^T(\mathbf{X}^T \mathbf{1}) &= \text{vec}(\mathbf{e}_i \mathbf{1}^T)^T \bar{x}
\end{align*}
\]
Comparing (13a) and (17) shows that
\[
a_0 = \sum_{i=1}^{M+K} \frac{c_i v_i}{v_i^T \bar{x}^T i} + \epsilon,
\]
where
\[
\mathbf{v}_i = \text{vec}(\mathbf{1} \mathbf{e}_i^T).
\]
Similarly, (13b) with (18) reveals that
\[
\mathbf{p}_i = \text{vec}(\mathbf{e}_i \mathbf{1}^T).
\]
Replacing $X$ in (17) by $W \circ X$ yields
\[ e_i^T (W \circ X) 1 = \text{vec}(1e_i^T)^T (\text{vec}(W) \circ \text{vec}(X)) = \text{vec}(1e_i^T) \circ W^T \hat{x} \] 
(22)

Let $W = W_Q + W_q$ with $W_Q = \begin{pmatrix} \tilde{w}^{(m,n)} & 0_{M \times K} \\ 0_{K \times M} & 0_{K \times K} \end{pmatrix}$ and $W_q = \begin{pmatrix} w^{(m,n)}_0 & 0_{M \times K} \\ 0_{K \times M} & 0_{K \times K} \end{pmatrix}$. Then putting this into (22) gives
\[ e_i^T (W \circ X) 1 = \text{vec}(1e_i^T) \circ W_Q^T \hat{x} + \text{vec}(1e_i^T) \circ W_q^T \hat{x} \]
(23)

Since both $W^{(m,n)}$ and $W^{(k,n)}$ in $W_q$ are fixed matrices and $W^{(m,k)}$ in $W_Q$ contains $X^{(k,n)}$, we have
\[ \frac{1}{2} \hat{x}^T Q_i \hat{x} = \text{vec}(1e_i^T) \circ W_q^T \hat{x} \] 
(24)

Therefore, we obtain
\[ Q_i = \text{vec}(1e_i^T) \circ W_q. \]

Due to the structure of $W_Q$, $\frac{1}{2} \hat{x}^T Q_i \hat{x} = 0$ and $Q_i = 0$ for $i > M$; otherwise, it holds
\[ \frac{1}{2} \hat{x}^T Q_i \hat{x} = \text{vec}(1e_i^T) \circ W_q^T \hat{x} \] 
(25)

where $\tilde{e}_i$ is a matrix filled with zeros except for the $i$-th row which is equal to $e_i^T$. Now for any matrix $A$, $B$ and $X$, it can be verified that $8^{\circ}$ refers to the Kronecker Product.

\[ (A \otimes B) \text{vec}(X) = \text{vec}(AXB), \]

and therefore
\[ \text{vec}(E_i 1^M (X^{(k,n)} 1^N)^T) = (I^K \otimes (E_i 1^M (1^N)^T)) \text{vec}(X^{(k,n)} 1^N)^T. \]

Given this, it can be shown that
\[ B_i = \Pi_0 (I^K \otimes (E_i 1^M (1^N)^T)) \text{diag} \left( \text{vec}(E_i \Omega^{(m,k)}) \right), \]

where $\Pi_0$ is a permutation matrix such that $(\hat{x}^{(k,n)})^T \Pi_0 = \text{vec}((X^{(k,n)})^T)^T$. Now, let $\Pi_0$ and $\Pi_0$ be permutation matrices such that
\[ \hat{x} = \begin{pmatrix} \Pi_0 \cdot \hat{x}^{(m,n)} \\ \Pi_0 \cdot \hat{x}^{(k,n)} \end{pmatrix}. \]

Considering this, we can rewrite (25) as
\[ \frac{1}{2} \hat{x}^T \cdot \begin{pmatrix} T_i & 0 \\ T_i^T & 0 \end{pmatrix} \cdot \hat{x} \]
(26)

Thus, we finally obtain
\[ \hat{q}_i = \begin{pmatrix} 0 \\ T_i^T \end{pmatrix}. \]
(28)

REFERENCES


